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AN INTRINSIC THEORY OF SHELLS

YOSHIYUKI YAMAMOTO
University of Tokyo, Bunkyo-ku, Tokyo

Abstract—Stresses in a shell can be deduced from stress function tensors which are regarded as an extension of the Maxwell–Morera stress function, while stress resultants and stress couples can be deduced from the shell stress function of Gol'denveizer. In this paper the relation between the stress function tensors and the shell stress function is investigated.

INTRODUCTION

IN MOST papers on the theory of shells, strains are deduced from displacements on the basis of the theory of continua, while displacements are determined by the equilibrium conditions for stress resultants and stress couples [1–3]. It is well known, however, that the introduction of stress functions is effective in analysing deformations of cylindrical and shallow shells [4, 5]. In Chien's intrinsic theory [6] the stress for equilibrium is determined directly by considering the conditions of equilibrium and compatibility. Gol'denveizer [7] introduced a system of stress functions by examining the general solution of the equilibrium equations. Recently Schaefer [8] and Günther [9] developed a theory of thin shells from the point of view of duality for displacements and stress functions. Langhaar [10] introduced stress functions in the membrane theory of shells.

In the analysis of shells, stress functions are so determined that the strain components derived from them satisfy the conditions of compatibility. Washizu [11] and Truesdell [12, 13] introduced stress functions from the principle of virtual work and the conditions of compatibility by the method of Lagrangean multipliers. This indicates the existence of an important relation between stress functions and the corresponding condition of compatibility [14].

In the present paper stress functions of a shell will be investigated on the basis of the general theory of continua, and the relation between stress functions will be clarified. It is assumed that the shell is subjected to neither body forces nor surface tractions. The theory is effective not only for thin shells but also for shells of finite thickness. The corresponding conditions of compatibility will also be derived.

1. MATHEMATICAL PRELIMINARIES

The configuration of a shell can be determined with reference to the Cartesian coordinates $x^i(i, j, k, l, h = \mathrm{I, II, III})$. The curvilinear coordinates $z^r(r, s, t, u, v, w = 1, 2, 3)$ are so introduced that the middle surface Π coincides with the coordinate surface $z^3 = 0$ and the z^3-curve ($z^1 = \text{const}$, $z^2 = \text{const}$) is normal to the surface Π. On the surface Π, $z^\rho(\rho, \sigma, \tau, \omega = 1, 2)$ is regarded as a two-dimensional curvilinear coordinate. The following relations are valid between the differentials, $\mathrm{d}x^i$ and $\mathrm{d}z^r$, of the coordinates;

$$\mathrm{d}x^i = A^i_r\,\mathrm{d}z^r \qquad \mathrm{d}z^r = A^r_i\,\mathrm{d}x^i, \tag{1}$$

$$A_r^i = \frac{\partial x^i}{\partial z^r} \equiv x^i_{,r} \qquad A_i^r = \frac{\partial z^r}{\partial x^i} \equiv z^r_{,i}, \tag{2}$$

where the summation convention is used and subscripts after commas indicate ordinary differentiation with respect to appropriate coordinates. Without loss of generality, it can be assumed that

$$x^i = x^{0i}(z^1, z^2) + p^i(z^1, z^2)z^3, \tag{3}$$

where p^i is the unit normal vector on Π:

$$p^i p^i = 1.$$

Then A_r^i can be written as follows;

$$A_r^i = A_r^{0i} + z^3 B_r^i, \tag{4}$$

where

$$\left.\begin{aligned} &A_\rho^{0i} = A_\rho^i(z^3 = 0) = x^{0i}_{,\rho}, \qquad A_3^{0i} = A_3^i(z^3 = 0) = p^i, \\ &B_\rho^i = p^i_{,\rho} \qquad B_3^i = 0 \qquad (\rho, \sigma, \tau, \omega = 1, 2) \end{aligned}\right\} \tag{5}$$

From the assumption it follows that

$$A_3^{0i} A_\rho^{0i} = 0 \qquad A_3^{0i} B_\rho^i = 0. \tag{6}$$

The fundamental tensor g_{rs} of the curvilinear coordinate is given by

$$\left.\begin{aligned} &g_{\rho\sigma} = A_\rho^i A_\sigma^i = g_{\rho\sigma}^0 + 2z^3 h_{\rho\sigma} + (z^3)^2 l_{\rho\sigma} \\ &g_{3\sigma} = g_{\sigma 3} \qquad g_{33} = 1, \end{aligned}\right\} \tag{7}$$

where

$$\left.\begin{aligned} &g_{\rho\sigma}^0 = g_{\rho\sigma}(z^3 = 0) = A_\rho^{0i} A_\sigma^{0i} \\ &h_{\rho\sigma} = B_{(\rho}^i A_{\sigma)}^i \qquad l_{\rho\sigma} = B_\rho^i B_\sigma^i. \end{aligned}\right\} \tag{8}$$

In this paper the parentheses ( ) used with indices indicate the mean with respect to indices,

$$B_{(\rho}^i A_{\sigma)}^{0\,i} = \tfrac{1}{2}(B_\rho^i A_\sigma^{0i} + B_\sigma^i A_\rho^{0i}).$$

It can be shown that

$$A_r^{0i} A_i^s = \begin{cases} e_{\rho\tau} e^{\sigma\omega} (\delta_\omega^\tau + z^3 h_\omega^\tau) \sqrt{(g^0/g)} \equiv \Lambda_\rho^\sigma & r = \rho \quad s = \sigma \\ 1 & r = s = 3 \\ 0 & \text{otherwise}, \end{cases} \tag{9}$$

where

$$\frac{1}{\sqrt{(g^0)}} e_{\rho\tau} = \sqrt{(g^0)} e^{\rho\tau} = \begin{cases} 1 & (\rho \quad \tau) = (1 \quad 2) \\ -1 & (\rho \quad \tau) = (2 \quad 1) \\ 0 & \text{otherwise}, \end{cases} \tag{10}$$

$$g^0 = |g^0_{\rho\sigma}| \qquad g = |g_{rs}| = [1+2z^3H+(z^3)^2R]^2 g^0 \tag{11}$$

$$\left.\begin{aligned} H &= \tfrac{1}{2}h^\rho_\rho \qquad & R &= \tfrac{1}{2}[(h^\rho_\rho)^2 - h^\rho_\sigma h^\sigma_\rho] \\ h^\rho_\sigma &= g^{0\rho\tau}h_{\tau\sigma} \qquad & g^{0\rho\tau}g^0_{\tau\sigma} &= \delta^\rho_\sigma. \end{aligned}\right\} \tag{12}$$

Let $t = t(z^1, z^2)$ be the thickness of the shell. Then the surface boundaries Π_+ and Π_- can be given by $z^3 = \pm t/2$. Without loss of generality, it can be assumed that Π and $\Pi_\pm$ are smooth surfaces.

The covariant derivative of a tensor T^{rs}_t with respect to z^u is defined by

$$T^{r\ s}_{t\ ;u} = T^{r\ s}_{t,u} + \begin{Bmatrix} r \\ r'u \end{Bmatrix} T^{r's}_t + \begin{Bmatrix} s \\ s'u \end{Bmatrix} T^{rs'}_t - \begin{Bmatrix} t' \\ tu \end{Bmatrix} T^{rs}_{t'}, \tag{13}$$

where $\begin{Bmatrix} r \\ st \end{Bmatrix}$ is Christoffel's symbol:

$$\begin{Bmatrix} r \\ st \end{Bmatrix} = \begin{Bmatrix} r \\ ts \end{Bmatrix} = g^{ru}(g_{u(s,t)} - \tfrac{1}{2}g_{st,u}) = A^r_i A^i_{s,t} = -A^r_{i,t}A^i_s \tag{14}$$

$$g^{ru}g_{us} = \delta^r_s.$$

As for a tensor defined on Π, two kinds of covariant differentiation can be defined. Let T^{0rs}_t be a generic tensor on Π. The two-dimensional covariant derivative of T^{0rs}_t with respect to z^ω is given by the equation of the following form:

$$T^{03\sigma}_{\tau/\omega} = T^{03\sigma}_{\tau,\omega} + \begin{Bmatrix} \sigma \\ \sigma'\omega \end{Bmatrix}^0 T^{03\sigma'}_\tau - \begin{Bmatrix} \tau' \\ \tau\omega \end{Bmatrix}^0 T^{03\sigma}_{\tau'}, \tag{15}$$

where $\begin{Bmatrix} \rho \\ \sigma\omega \end{Bmatrix}^0$ is Christoffel's symbol for Π,

$$\begin{Bmatrix} \rho \\ \sigma\omega \end{Bmatrix}^0 = \begin{Bmatrix} \rho \\ \sigma\omega \end{Bmatrix}(z^3 = 0) = g^{0\rho\omega}(g^0_{\omega(\sigma,\tau)} - \tfrac{1}{2}g^0_{\sigma\tau,\omega}). \tag{16}$$

The three-dimensional covariant derivative is defined as follows:

$$T^{03\sigma}_{\tau\,//\omega} = T^{03\sigma}_{\tau/\omega} - h_{\rho\omega}T^{0\rho\sigma}_\tau + h^\sigma_\omega T^{033}_\tau + h_{\tau\omega}T^{03\sigma}_3. \tag{17}$$

Since

$$\begin{Bmatrix} 3 \\ \rho\omega \end{Bmatrix}(z^3 = 0) = -h_{\rho\omega}, \qquad \begin{Bmatrix} \rho \\ 3\omega \end{Bmatrix}(z^3 = 0) = h^\rho_\omega, \qquad \begin{Bmatrix} 3 \\ 3\omega \end{Bmatrix} = 0, \tag{18}$$

it can easily be seen that

$$T^{r\ s}_{t\ ;\omega}(z^3 = 0) = T^{0rs}_{t//\omega},$$

if

$$T^{rs}_t(z^3 = 0) = T^{0rs}_t.$$

If $T^{0rs}_{t//3}$ is appropriately assumed, $T^{0rs}_{t//\rho\sigma}$ can also be defined.

2. STRESS FUNCTION TENSOR FOR SHELLS

When the elastic body is not subjected to any body forces, the condition of equilibrium in the body referred to Cartesian coordinates is

$$\sigma^{ij}_{,j} = 0.$$

Any stress distribution for equilibrium can be represented by using an appropriate stress function tensor ψ^{ikjl} which is considered as an extension of the Maxwell–Morera stress functions:

$$\sigma^{ij} = \psi^{ikjl}_{,kl} = e^{iki'} e^{jlj'} \psi_{i'j'} \tag{1}$$

where

$$\left.\begin{aligned} \psi^{ikjl} &= \psi^{jlik} = -\psi^{kijl} = e^{iki'} e^{jlj'} \psi_{i'j'} \\ \psi_{i'j'} &= \psi_{j'i'} = \tfrac{1}{4} e_{iki'} e_{jlj'} \psi^{ikjl} \end{aligned}\right\} \tag{2}$$

$$e^{ijk} = e_{ijk} = \begin{cases} 1 & (ijk) \text{ is an even permutation of (I II III)} \\ -1 & (ijk) \text{ is an odd permutation of (I II III)} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

With reference to curvilinear coordinates, the condition of equilibrium becomes

$$\sigma^{rs}{}_{;s} = 0, \qquad \sigma^{rs} = \sigma^{ij} A^{rs}_{ij}, \tag{4}$$

where

$$A^{rs\ldots}_{ij\ldots} = A^r_i A^s_j \ldots . \tag{5}$$

The stress can be written as

$$\sigma^{rs} = \psi^{rtsu}{}_{;tu} = e^{rtv} e^{suw} \psi_{vw;tu}, \tag{6}$$

where

$$\left.\begin{aligned} \psi^{rtsu} &= A^{rstu}_{ikjl} \psi^{ikjl} = e^{rtv} e^{suw} \psi_{vw} \\ \psi_{vw} &= A^{ij}_{vw} \psi_{ij} = \tfrac{1}{4} e_{rtv} e_{suw} \psi^{rtsu} \end{aligned}\right\} \tag{7}$$

$$\sqrt{(g)} e^{rst} = \frac{1}{\sqrt{(g)}} e_{rst} = \begin{cases} 1 & (rst) \text{ is an even permutation of (123)} \\ -1 & (rst) \text{ is an odd permutation of (123)} \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Since

$$e^{rtv} e^{suw} (\psi_{vw;tu} - X_{(v;w);tu}) = e^{rtv} e^{suw} \psi_{vw;tu},$$

it then follows that

LEMMA 1. *Two stress function tensors*

$$\psi_{vw} \quad \text{and} \quad \bar{\psi}_{vw} = \psi_{vw} - X_{(v;w)} \tag{9}$$

furnish the same stress distribution, where X_v is an arbitrary vector.

Now the following differential equations can be solved:

$$\left.\begin{aligned} \bar{\psi}_{33} &= \psi_{33}-X_{3;3} = \psi_{33}-X_{3,3} = 0 \\ \bar{\psi}_{3\sigma} &= \psi_{3\sigma}-\tfrac{1}{2}[X_{3,\sigma}+X_{\sigma,3}-2(h_\sigma^\rho+3l_\sigma^\rho 2^3+\cdots)X_3] = 0 \end{aligned}\right\} \tag{10}$$

or

$$\left.\begin{aligned} X_{3,3} &= \psi_{33} \\ X_{\sigma,33}-2[(h_\sigma^\rho+3l_\sigma^\rho z^3+\cdots)X_\rho]_{,3} &= 2\psi_{3\sigma,3}-\psi_{33,\sigma}. \end{aligned}\right\}$$

By the use of a vector X_v thus obtained, a stress function tensor $\bar{\psi}_{vw}$ can be determined, and can be adopted instead of ψ_{vw}. Then it follows that [16]

THEOREM 1. *A stress function tensor ψ_{rs} can be chosen so that*

$$\psi_{3s} = \psi_{r3} = 0. \tag{11}$$

The corresponding stress is given by

$$\left.\begin{aligned} \sigma^{\rho\sigma} &= 4e^{\rho\tau 3}e^{\sigma\omega 3}\psi_{[\tau[\omega;3]3]} \\ \sigma^{3\sigma} &= -2e^{\rho\tau 3}e^{\sigma\omega 3}\psi_{[\tau[\omega;3]\rho]} \\ \sigma^{33} &= e^{\rho\tau 3}e^{\sigma\omega 3}\psi_{[\tau[\omega;\sigma]\rho]}. \end{aligned}\right\} \tag{12}$$

In this paper the brackets [ ] used with indices indicate the mixing of indices;

$$\left.\begin{aligned} \psi_{\tau[\omega;\sigma]\rho} &= \tfrac{1}{2}(\psi_{\tau\omega;\sigma\rho}-\psi_{\tau\sigma;\omega\rho}) \\ \psi_{[\tau[\omega;\sigma]\rho]} &= \tfrac{1}{4}(\psi_{\tau\omega;\sigma\rho}-\psi_{\tau\sigma;\omega\rho}-\psi_{\rho\omega;\sigma\tau}+\psi_{\rho\sigma;\omega\tau}). \end{aligned}\right\}$$

3. THE STRESS RESULTANT AND COUPLE

Without loss of generality, it can be assumed that the shell under consideration is simply connected. Let O and P be a fixed point and a generic point on the middle surface Π, respectively. Let the points on the surface boundaries Π_+ and Π_- corresponding to O and P be denoted by O_+, O_-, P_+, and P_-, respectively:

$$x^i(P_\pm) = x^i(P) \pm p^i(P)t/2.$$

It can be easily verified that

$$\psi^{i123}+\psi^{i231}+\psi^{i312} = 0$$

or

$$\psi^{ii'jk}+\psi^{ijki'}+\psi^{iki'j} = 0.$$

This relation can be rewritten as,

$$e_{khi}\psi^{ii'jh} = e_{khi}\tfrac{1}{2}(\psi^{ii'jh}-\psi^{hi'ji}) = \tfrac{1}{2}e_{khi}\psi^{ihji'} \tag{1}$$

The resultant, and the moment about the axes through the point O parallel to the axes x^k, of stresses associated with surface portion S in the shell can be expressed as a line integral along the boundary ∂S of S by virtue of the Green–Stokes formula [15, 14]:

$$\iint_S \sigma^{ij} n_j \, dS = \iint_S \psi^{ii'jh}{}_{,i'h} n_j \, dS = \tfrac{1}{2} \int_{\partial S} \psi^{ii'jh}{}_{,i'} e_{jhl} \, dx^l = e^{ijk} \int_{\partial S} \psi_{l[k,j]} \, dx^l, \tag{2a}$$

$$\begin{aligned} e_{khi} \iint_S (x^h - x^h(O)) \sigma^{ij} n_j \, dS &= e_{khi} \iint_S (x^h - x^h(O)) \psi^{ii'jj'}{}_{,i'j'} n_j \, dS \\ &= e_{khi} \iint_S \{[(x^h - x^h(O)) \psi^{ii'jj'}{}_{,i'}]_{,j'} - \psi^{ii'jh}{}_{,i'}\} n_j \, dS \\ &= e_{khi} \iint_S \{[(x^h - x^h(O)) \psi^{ii'jj'}{}_{,i'}]_{,j'} - \tfrac{1}{2} \psi^{ihjj'}{}_{,j'}\} n_j \, dS \\ &= -e_{ihk} \tfrac{1}{2} \int_{\partial S} [(x^h - x^h(O)) \psi^{ii'jj'}{}_{,i'} - \tfrac{1}{2} \psi^{ihjj'}] e_{jj'l} \, dx^l \\ &= -\int_{\partial S} [2(x^h - x^h(O)) \psi_{l\,[k,h]} - \psi_{lk}] \, dx^l, \end{aligned} \tag{2b}$$

where n_j is the unit normal vector on S.

Since the surface boundaries of the shell are free of external forces, the resultant and the moment of stresses associated with a portion of the surface boundaries vanish. Hence the functions

$$\Psi^i(P_\pm) = \tfrac{1}{2} \int_{O_\pm}^{P_\pm} \psi^{ii'jk}{}_{,i'} e_{jkl} \, dx^l = e^{ijk} \int_{O_\pm}^{P_\pm} \psi_{l[k,j]} \, dx^l, \tag{3a}$$

$$\begin{aligned} 2\Psi_k(P_\pm) &= \tfrac{1}{2} \int_{O_\pm}^{P_\pm} [(x^h - x^h(O)) \psi^{ii'jj'}{}_{,i'} - \tfrac{1}{2} \psi^{ihjj'}] e_{ihk} e_{jj'l} \, dx^l \\ &= \int_{O_\pm}^{P_\pm} [2(x^j - x^j(O)) \psi_{l[k,j]} - \psi_{lk}] \, dx^l \end{aligned} \tag{3b}$$

are independent of the paths of integration on $\Pi_\pm$, and are regarded as point functions on $\Pi_\pm$. These functions can be easily rewritten as,

$$\Psi_{jk}(P_\pm) = -\Psi_{kj}(P_\pm) = e_{ijk} \Psi(P_\pm) = 2 \int_{O_\pm}^{P_\pm} \psi_{l[k,j]} \, dx^l \tag{4a}$$

$$2\Psi_k(P_\pm) = \int_{O_\pm}^{P_\pm} [2(x^j - x^j(O)) \psi_{l[k,j]} - \psi_{lk}] \, dx^l = (x^j - x^j(O)) \Psi_{jk} - \int_{O_\pm}^{P_\pm} (\Psi_{lk} + \psi_{lk}) \, dx^l. \tag{4b}$$

Let ψ_k be such a tensor defined on $\Pi_\pm$ that

$$\psi_k(P_\pm) = (x^j(P_\pm) - x^j(O)) \Psi_{jk}(P_\pm) - 2\Psi_k(P_\pm). \tag{5}$$

It then follows from (4b) that

$$\psi_k(P_\pm) = \int_{O_\pm}^{P_\pm} (\Psi_{lk} + \psi_{lk}) \, dx^l \tag{6a}$$

or

$$\psi_r(P_\pm) = A_r^k(P_\pm) \int_{O_\pm}^{P_\pm} A_k^s (\Psi_{ts} + \psi_{ts}) \, dz^t, \tag{6b}$$

where

$$\psi_r = A_r^k \psi_k, \qquad \Psi_{rs} = -\Psi_{sr} = A_{rs}^{jk} \Psi_{jk}, \qquad \psi_{rs} = \psi_{sr} = A_{rs}^{jk} \psi_{jk}.$$

When σ_{rs} vanishes on a portion of the shell, Ψ_{rs}, ψ_{rs}, and ψ_r can be defined there, and the relations hold

$$\psi_{[r;s]} = \Psi_{sr} \qquad \psi_{(r;s)} = \psi_{sr}. \tag{7}$$

Now we define the vector functions Ω^i and Φ_k on Π such that

$$\Omega^i(\mathrm{P}) = e^{ijk}\Big(\int_{\mathrm{O}_-}^{\mathrm{P}_-} + \int_{\mathrm{P}_-}^{\mathrm{P}_+} - \int_{\mathrm{O}_+}^{\mathrm{P}_+} - \int_{\mathrm{O}_-}^{\mathrm{O}_+}\Big)\psi_{l[k,j]}\,\mathrm{d}x^l, \tag{8a}$$

$$\begin{aligned}\Phi_k(\mathrm{P}) &= -\Big(\int_{\mathrm{O}_-}^{\mathrm{P}_-} + \int_{\mathrm{P}_-}^{\mathrm{P}_+} - \int_{\mathrm{O}_+}^{\mathrm{P}_+} - \int_{\mathrm{O}_-}^{\mathrm{O}_+}\Big)[2(x^j - x^j(\mathrm{P}))\psi_{l[k,j]} - \psi_{lk}]\,\mathrm{d}x^l \\ &= -\Big(\int_{\mathrm{O}_-}^{\mathrm{P}_-} + \int_{\mathrm{P}_-}^{\mathrm{P}_+} - \int_{\mathrm{O}_+}^{\mathrm{P}_+} - \int_{\mathrm{O}_-}^{\mathrm{O}_+}\Big)[2(x^j - x^j(\mathrm{O}))\psi_{l[k,j]} - \psi_{lk}]\,\mathrm{d}x^l \\ &\quad + e_{ijk}(x^j(\mathrm{P}) - x^j(\mathrm{O}))\Omega^i(\mathrm{P}),\end{aligned} \tag{8b}$$

where the integrals $\int_{\mathrm{O}_+}^{\mathrm{P}_+}\ldots$ and $\int_{\mathrm{O}_-}^{\mathrm{P}_-}\ldots$ are taken along certain paths on $\Pi_\pm$ and the integrals $\int_{\mathrm{O}_-}^{\mathrm{O}_+}\ldots$ and $\int_{\mathrm{P}_-}^{\mathrm{P}_+}\ldots$ along the z^3-curves.

The stress resultant $N^{r\rho}$ and the stress couple M_r^ρ can be defined on Π: Let P′ be a neighboring point of P on Π. The relations hold:

$$A_r^{0i}N^{r\rho}(\mathrm{P})e_{\rho\sigma}\,\mathrm{d}z^\sigma = \int_{\mathrm{P}_-}^{\mathrm{P}_+} A_s^i\sigma^{s\rho}e_{\rho\sigma 3}\,\mathrm{d}z^\rho\,\mathrm{d}z^3 = e^{ijk}\Big(\int_{\mathrm{P}_-}^{\mathrm{P}'_-} + \int_{\mathrm{P}'_-}^{\mathrm{P}'_+} - \int_{\mathrm{P}_+}^{\mathrm{P}'_+} - \int_{\mathrm{P}_-}^{\mathrm{P}_+}\Big)\psi_{l[k,j]}\,\mathrm{d}x^l \tag{9a}$$

$$\begin{aligned}A_k^{0r}M_r^\rho(\mathrm{P})e_{\rho\sigma}\,\mathrm{d}z^\sigma &= e_{khi}\int_{\mathrm{P}_-}^{\mathrm{P}_+}(x^h - x^h(\mathrm{P}))A_s^i\sigma^{s\rho}e_{\rho\sigma 3}\,\mathrm{d}z^\sigma\,\mathrm{d}z^3 \\ &= -\Big(\int_{\mathrm{P}_-}^{\mathrm{P}'_-} + \int_{\mathrm{P}'_-}^{\mathrm{P}'_+} - \int_{\mathrm{P}_+}^{\mathrm{P}'_+} - \int_{\mathrm{P}_-}^{\mathrm{P}_+}\Big)\{2(x^h - x^h(\mathrm{P}))\psi_{l[k,h]} - \psi_{lk}\}\,\mathrm{d}x^l,\end{aligned} \tag{9b}$$

where

$$z^\sigma(\mathrm{P}') = z^\sigma(\mathrm{P}) + \mathrm{d}z^\sigma \qquad z^3(\mathrm{P}') = 0$$

$$A_i^{0r} = A_i^r(z^3 = 0) \qquad e^0{}_{rst} = e_{rst}(z^3 = 0) \qquad e_{\rho\sigma} = e^0{}_{\rho\sigma 3}.$$

Then the expressions for $N^{r\rho}$ and M_r^ρ become

$$N^{r\rho} = A_i^{0r}\int A_s^i\sigma^{s\rho}\sqrt{\Big(\frac{g}{g^0}\Big)}\,\mathrm{d}z^3 \tag{10a}$$

$$M_\tau^\rho = e^0_{\tau 3\sigma}A_i^{0\sigma}\int z^3 A_s^i\sigma^{s\rho}\sqrt{\Big(\frac{g}{g^0}\Big)}\,\mathrm{d}z^3 \tag{10b}$$

$$M_3^\rho = 0. \tag{10c}$$

For the sake of convenience, it will be assumed that

$$N^{r3} = M_r^3 = 0.$$

Then N^{rs} and M_r^s are considered as tensors defined on Π. From (8) it follows that

$$\begin{aligned}\Omega^i(\mathrm{P}') - \Omega^i(\mathrm{P}) &\equiv \Omega^i{}_{,l}A_\sigma^{0l}\,\mathrm{d}z^\sigma \\ &= e^{ijk}\Big(\int_{\mathrm{P}_-}^{\mathrm{P}'_-} + \int_{\mathrm{P}'_-}^{\mathrm{P}'_+} + \int_{\mathrm{P}'_+}^{\mathrm{P}_+} + \int_{\mathrm{P}_+}^{\mathrm{P}_-}\Big)\psi_{l[k,j]}\,\mathrm{d}x^l \\ &= A_r^{0i}N^{r\rho}e_{\rho\sigma}\,\mathrm{d}z^\sigma\end{aligned} \tag{11a}$$

$$\begin{aligned}\Phi_k(\mathrm{P}')-\Phi_k(\mathrm{P}) &\equiv \Phi_{k,l}A^{0l}_\sigma\,\mathrm{d}z^\sigma\\ &= -\left(\int_{\mathrm{P}_-}^{\mathrm{P}'_-}+\int_{\mathrm{P}'_-}^{\mathrm{P}'_+}+\int_{\mathrm{P}'_+}^{\mathrm{P}_+}+\int_{\mathrm{P}_+}^{\mathrm{P}_-}\right)[2(x^h-x^h(\mathrm{O}))\psi_{l[k,h]}-\psi_{lk}]\,\mathrm{d}x^l\\ &\qquad +e_{ijk}\Omega^iA^{0j}_\sigma\,\mathrm{d}z^\sigma+e_{ijk}(x^j(\mathrm{P})-x^j(\mathrm{O}))\Omega^i{}_{,l}A^{0l}_\sigma\,\mathrm{d}z^\sigma \qquad (11\mathrm{b})\\ &= A^{0r}_kM^\rho_re_{\rho\sigma}\,\mathrm{d}z^\sigma+e_{ijk}\Omega^iA^{0j}_\sigma\,\mathrm{d}z^\sigma\end{aligned}$$

or

$$\left.\begin{aligned}A^{0i}_rN^{r\rho} &= \Omega^i{}_{,l}A^{0l}_\sigma e^{\rho\sigma}\\ A^{0r}_kM^\rho_r+e_{ijk}\Omega^iA^{0j}_\sigma e^{\rho\sigma} &= \Phi_{k,l}A^{0l}_\sigma e^{\rho\sigma},\end{aligned}\right\}$$

where

$$e^{0rst} = e^{rst}(z^3=0), \qquad e^{\rho\sigma} = e^{0\rho\sigma3}.$$

Hence there holds

THEOREM 2. *The stress resultant $N^{r\tau}$ and the stress couple M^τ_r can be expressed in terms of $\Omega^r = A^{0r}_i\Omega_i$ and $\Phi_r = A^{0i}_r\Phi_i$ as:*

$$\left.\begin{aligned}N^{r\tau} &= \Omega^r_{/\!/\sigma}\,e^{\tau\sigma}\\ M^\tau_r &= (\Phi_{r/\!/\sigma}-e^0_{s\sigma r}\Omega^s)e^{\tau\sigma}\end{aligned}\right\} \qquad (12)$$

These satisfy the equilibrium condition:

$$\left.\begin{aligned}N^{r\tau}_{/\!/\tau} &= \Omega^r_{/\!/\sigma\tau}e^{0\tau\sigma3} = 0\\ M^\tau_{\rho/\!/\tau} &= -(\Phi_{\rho/\!/\sigma\tau}-e^0_{3\sigma\rho}\Omega^3_{/\!/\tau})e^{\sigma\tau} = e^0_{3\sigma\rho}N^{3\sigma} = e_{\sigma\rho}N^{3\sigma}\\ M^\tau_{3/\!/\tau} &\equiv -h^\rho_\tau M^\tau_\rho = e^0_{\rho\sigma3}N^{\rho\sigma} = \sqrt{(g^0)}(N^{12}-N^{21}).\end{aligned}\right\} \qquad (13)$$

From (10c) and (12) it is obvious that

$$\Omega^\rho = e^{\rho\sigma}\Phi_{3/\!/\sigma}. \qquad (14)$$

Then Ω^r and Φ_r are Gol'denveizer's stress functions. These are also equivalent to the stress functions introduced by Schaefer [8] and Günther [9].

Instead of Ω^i, the following stress function Ω_{jk} may be conveniently used:

$$\left.\begin{aligned}\Omega_{jk} &= e_{ijk}\Omega^i = \left(\int_{\mathrm{O}_-}^{\mathrm{P}_-}+\int_{\mathrm{P}_-}^{\mathrm{P}_+}-\int_{\mathrm{O}_+}^{\mathrm{P}_+}-\int_{\mathrm{O}_-}^{\mathrm{O}_+}\right)2\psi_{l[k,j]}\,\mathrm{d}x^l\\ \Omega^i &= \tfrac{1}{2}e^{ijk}\Omega_{jk},\ \Omega_{st} = A^{0ik}_{st} = e^0_{rst}\Omega^r.\end{aligned}\right\} \qquad (15)$$

The relation (11a) can be rewritten as,

$$\Omega_{jk,l}A^{0l}_\sigma = e_{ijk}A^{0i}_rN^{r\rho}e_{\rho\sigma}.$$

An alternative expression for stress couples is defined by

$$M^{\rho\sigma} = e^{\rho\tau}M^\sigma_\tau \qquad M^{3\sigma} = M^{\rho3} = M^{33} = 0. \qquad (16)$$

It then follows that

COROLLARY 2.1. *Stress resultants and couples can be written in terms of* Ω_{rs} *and* Φ_r *as:*

$$\left.\begin{aligned} N^{\rho\sigma} &= \tfrac{1}{2}e^{0\rho st}e^{\sigma\omega}\Omega_{st/\!/\omega} = e^{\rho\tau}e^{\sigma\omega}\Omega_{\tau 3/\!/\omega} \\ N^{3\sigma} &= \tfrac{1}{2}e^{03\rho\tau}e^{\sigma\omega}\Omega_{\rho\tau/\!/\omega} = \tfrac{1}{2}e^{\rho\tau}e^{\sigma\omega}\Omega_{\rho\tau/\!/\omega} \\ M^{\rho\sigma} &= e^{\rho\tau}e^{\sigma\omega}(\Phi_{\tau/\!/\omega}+\Phi_{\tau\omega}) \end{aligned}\right\} \tag{17}$$

$$\Phi_{3/\!/\omega}+\Omega_{3\omega} = 0. \tag{18}$$

By virtue of (3) and (8), the functions Ω^i, Ω_{jk}, and Φ_k can be written in the form,

$$\Omega^i(\mathrm{P}) = \Psi^i(\mathrm{P}_-)-\Psi^i(\mathrm{P}_+)+e^{ijk}\Big(\int_{\mathrm{P}_-}^{\mathrm{P}_+} - \int_{\mathrm{O}_-}^{\mathrm{O}_+}\Big)\psi_{l[k,j]}\,\mathrm{d}x^l \tag{19a}$$

$$\Omega_{jk}(\mathrm{P}) = \Psi_{jk}(\mathrm{P}_-)-\Psi_{jk}(\mathrm{P}_+)+\Big(\int_{\mathrm{P}_-}^{\mathrm{P}_+} - \int_{\mathrm{O}_-}^{\mathrm{O}_+}\Big)2\psi_{l[k,j]}\,\mathrm{d}x^l \tag{19b}$$

$$\begin{aligned}\Phi_k(\mathrm{P}) = {} & -2\Psi_k(\mathrm{P}_-)+2\Psi_k(\mathrm{P}_+) \\ & -\Big(\int_{\mathrm{P}_-}^{\mathrm{P}_+} - \int_{\mathrm{O}_-}^{\mathrm{O}_+}\Big)[2(x^j-x^j(\mathrm{O}))\psi_{l[k,j]}-\psi_{lk}]\,\mathrm{d}x^l \\ & +\Omega_{jk}(x^j(\mathrm{P})-x^j(\mathrm{O})). \end{aligned} \tag{20}$$

From (17), (19) and (20), it follows that

$$N^{r\sigma} = \tfrac{1}{2}e^{0rst}e^{\sigma\omega}A^{0ijk}_{st\omega}\frac{\partial\Omega_{ij}(P)}{\partial x^k(P)}$$

$$M^{\rho\sigma} = e^{\rho\tau}e^{\sigma\omega}A^{0kj}_{\tau\omega}\left\{\frac{\partial}{\partial x^j(\mathrm{P})}\Big[-2\Psi_k(\mathrm{P})+\ \ldots\Big(\int_{\mathrm{P}_-}^{\mathrm{P}_+} - \int_{\mathrm{O}_-}^{\mathrm{O}_+}\Big)\ldots\Big]+(x^l(\mathrm{P})-x^l(\mathrm{O}))\frac{\partial\Omega_{lk}(\mathrm{P})}{\partial x^j(\mathrm{P})}\right\}.$$

Then it can be seen that

LEMMA 2. *Stress resultants and couples are independent of the integrals between* O_- *and* O_+ *in the expressions for* Ω_{jk} *and* Φ_k. *Accordingly, such terms can be omitted when* Ω_{jk} *and* Φ_k *are used as stress functions.*

Hereafter such integrals will be omitted in the expressions for Ω_{jk} and Φ_k. It then follows that

$$\Omega_{jk}(\mathrm{P}) = \Psi_{jk}(\mathrm{P}_-)-\Psi_{jk}(\mathrm{P}_+)+\int_{\mathrm{P}_-}^{\mathrm{P}_+} 2\Psi_{l[k,j]}\,\mathrm{d}x^l \tag{21a}$$

$$\begin{aligned}\Phi_k(\mathrm{P}) = {} & \psi_k(\mathrm{P}_-)-(x^j(\mathrm{P}_-)-x^j(\mathrm{O}))\Psi_{jk}(\mathrm{P}_-)-\psi_k(\mathrm{P}_+)+(x^j(\mathrm{P}_+)-x^j(\mathrm{O}))\Psi_{jk}(\mathrm{P}_+) \\ & +(\Psi_{jk}(\mathrm{P}_-)-\Psi_{jk}(\mathrm{P}_+))(x^j(\mathrm{P})-x^j(\mathrm{O}))-\int_{\mathrm{P}_-}^{\mathrm{P}_+}[2(x^j-x^j(\mathrm{P}))\psi_{l[k,j]}-\psi_{lk}]\,\mathrm{d}x^l \\ = {} & \psi_k(\mathrm{P}_-)-\psi_k(\mathrm{P}_+)-(x^j(\mathrm{P}_-)-x^j(\mathrm{P}))\Psi_{jk}(\mathrm{P}_-)+(x^j(\mathrm{P}_+)-x^j(\mathrm{P}))\Psi_{jk}(\mathrm{P}_+) \\ & -\int_{\mathrm{P}_-}^{\mathrm{P}_+} A^r_k(z^3\psi_{3r;3}-z^3\psi_{33;r}-\psi_{3r})\,\mathrm{d}z^3. \end{aligned} \tag{21b}$$

Now the stress functions Ω_{rs} and Φ_r are expressed in terms of Ψ_{rs} and ψ_r:

THEOREM 3. *Stress functions Ω_{rs} and Φ_r are given in terms of Ψ_{rs} and Ψ_r when $\psi_{3s} = \psi_{r3} = 0$:*

$$\left.\begin{aligned}\Omega_{\rho\sigma} &= -\Omega_{\sigma\rho} = \Lambda_{\rho\sigma}^{\rho'\sigma'}(\mathrm{P}_-)\Psi_{\rho'\sigma'}(\mathrm{P}_-) - \Lambda_{\rho\sigma}^{\rho'\sigma'}(\mathrm{P}_+)\Psi_{\rho'\sigma'}(\mathrm{P}_+) + \int_{\mathrm{P}_-}^{\mathrm{P}_+} \Lambda_{\rho\sigma}^{\rho'\sigma'} 2\psi_{3[\sigma';\rho]}\, dz^3 \\ \Omega_{\rho 3} &= -\Omega_{3\rho} = \Lambda_\rho^{\rho'}(\mathrm{P}_-)\Psi_{\rho'3}(\mathrm{P}_-) - \Lambda_\rho^{\rho'}(\mathrm{P}_+)\Psi_{\rho'3}(\mathrm{P}_+)\end{aligned}\right\} \tag{22}$$

$$\left.\begin{aligned}\Phi_\rho &= \Lambda_\rho^{\rho'}(\mathrm{P}_-)(\psi_{\rho'}(\mathrm{P}_-) + \tfrac{t}{2}\Psi_{3\rho'}(\mathrm{P}_-)) - \Lambda_\rho^{\rho'}(\mathrm{P}_+)(\psi_{\rho'}(\mathrm{P}_+) - \tfrac{t}{2}\Psi_{3\rho'}(\mathrm{P}_+)) \\ \Phi_3 &= \psi_3(\mathrm{P}_-) - \psi_3(\mathrm{P}_+),\end{aligned}\right\} \tag{23}$$

where

$$\Lambda_{\rho\sigma}^{\rho'\sigma'} = \Lambda_\rho^{\rho'}\Lambda_\sigma^{\sigma'}. \tag{24}$$

4. CONDITION OF COMPATIBILITY

The condition of compatibility can be deduced by the use of the principle of complementary virtual work: The work W done by the stress $\sigma^{ij} = \psi^{ikjl}{}_{,kl}$ has equilibrium in the strain ε_{ij} should vanish. Hence the relation holds

$$\begin{aligned}W &= \iiint \sigma^{ij}\varepsilon_{ij}\, dV = \iiint \psi^{ikjl}{}_{,kl}\varepsilon_{ij}\, dV \\ &= \iiint \varepsilon_{[i[j,l]k]}\psi^{ikjl}\, dV + I_S = 0,\end{aligned} \tag{1}$$

where dV is the volume element. The integral I_S is given by the surface integral taken over the whole surface of the shell;

$$I_S = \iint (-\varepsilon_{ij,k}\psi^{ikjl} + \varepsilon_{ij}\psi^{ikjl}{}_{,k})n_l\, dS, \tag{2}$$

where $n_l = n^l$ is the outward unit normal vector on the surface. From (3) of Section 3, it follows that

$$\begin{aligned}\Psi^i{}_{,l} &= \tfrac{1}{2}\psi^{ii'jk}{}_{,i'}e_{jkl} \\ 2\Psi_{k'l} &= [\tfrac{1}{2}(x^h - x^h(\mathrm{O}))\psi^{ii'jj'}{}_{,i'} - \tfrac{1}{4}\Psi^{ihjj'}]e_{ihk}e_{jj'l} \\ &= [(x^h - x^h(\mathrm{O}))\Psi^i{}_{,l} - \tfrac{1}{4}\psi^{ihjj'}e_{jj'l}]e_{ihk}.\end{aligned}$$

The stress function tensor Ψ^{ikjl} can be expressed in terms of Ψ^i and Ψ_k on the surface boundaries

$$\left.\begin{aligned}\psi^{ikjl}{}_{,k} &= \Psi^i{}_{,k}e^{jlk} \\ \psi^{ikjl} &= [-2e^{ikh}\Psi_{h,j'} + 2(x^{[k} - x^{[k}(\mathrm{O}))\Psi^{i]}{}_{,j'}]e^{jlj'}.\end{aligned}\right\} \tag{3}$$

Introducing (3) into (2), and using (5) of Section 3, the following relation can be derived:

$$
\begin{aligned}
I_S &= \left(\iint_{\Pi_+} + \iint_{\Pi_-}\right)\{-\varepsilon_{ji,k}[-2e^{ikh}\Psi_{h,l} + 2(x^{[k} - x^{[k}(\mathrm{O}))\Psi^{i]}{}_{,l}] + \varepsilon_{ij}\Psi^i{}_{,l}\}e^{jj'l}n'_{j'}\,\mathrm{d}S \\
&\qquad + [\text{integral on the edge boundary}] \\
&= \left(\iint_{\Pi_+} + \iint_{\Pi_-}\right)\{\varepsilon_{[j[i,h]l]}[2e^{ikh}\Psi_h - 2(x^k - x^k(\mathrm{O}))\tfrac{1}{2}e^{ijh}\Psi_{jh}] - \varepsilon_{i[j,l]}\Psi^i + \varepsilon_{j[i,k]}\delta^k_l\Psi^i \\
&\qquad - \varepsilon_{l[i,k]}\delta^k_j\Psi^i\}e^{jlj'}n_{j'}\,\mathrm{d}S \\
&\qquad + [\text{integral on the edge boundary}] \\
&= -\left(\iint_{\Pi_+} + \iint_{\Pi_-}\right)\varepsilon_{[j[i,k]l]}\psi_h e^{ikh}e^{jlj'}n_{j'}\,\mathrm{d}S \\
&\qquad + [\text{integral on the edge boundary}].
\end{aligned}
$$

Now the expression for W becomes

$$
\begin{aligned}
W \equiv \iiint \varepsilon_{[r[s;u]t]}\psi_{vw}e^{rtv}e^{suw}\,\mathrm{d}V + \left(\iint_{\Pi_+} + \iint_{\Pi_-}\right)\varepsilon_{[r[s;u]t]}\psi_v n_w e^{rtv}e^{suw}\,\mathrm{d}S& \\
+ [\text{integral on the edge boundary}] = 0.&
\end{aligned}
\tag{4}
$$

The condition that the work W vanishes for any ψ_{vw} and any ψ_v is the condition of compatibility. Since $\psi_{3\omega} = 0$, it follows that

THEOREM 4. *The condition of compatibility can be expressed as:*

$$R_{3\tau 3\omega} = 0 \qquad \textit{throughout the shell} \tag{5}$$

$$R_{rt[sun\omega]} = 0 \qquad \textit{on} \qquad \Pi_\pm \tag{6}$$

where

$$
\begin{aligned}
R_{rtsu} &= 4\varepsilon_{[r[s;u]t]} \\
&= \varepsilon_{rs;ut} - \varepsilon_{ru;st} - \varepsilon_{ts;ur} + \varepsilon_{tu;sr}.
\end{aligned}
\tag{7}
$$

Without loss of generality, it can be assumed that n_3 does not vanish on $\Pi_\pm$. Hence the following relations hold:

$$R_{rt\sigma\omega} = 0 \qquad \text{or} \qquad R_{3\tau 12} = R_{1212} = 0 \qquad \text{on} \qquad \Pi_\pm. \tag{8}$$

It can be easily verified that

$$R_{rs12;3} + R_{rs23;1} + R_{rs31;2} = 0$$

(Bianchi's equality) or

$$R_{3\sigma 12,3} + 2R_{3\sigma 3[1,2]} - R_{3\rho 12}\begin{Bmatrix}\rho\\3\sigma\end{Bmatrix} - 2R_{\rho\sigma 3[1}\begin{Bmatrix}\rho\\2]3\end{Bmatrix} - 2R_{3\rho 3[1}\begin{Bmatrix}\rho\\2]\sigma\end{Bmatrix} = 0,$$

$$R_{1212,3} - 2R_{123[2,1]} - R_{\rho 212}\begin{Bmatrix}\rho\\13\end{Bmatrix} - 2R_{1\rho 12}\begin{Bmatrix}\rho\\32\end{Bmatrix} - 2R_{r23[1}\begin{Bmatrix}r\\2]1\end{Bmatrix} - 2R_{1r3[1}\begin{Bmatrix}r\\2]2\end{Bmatrix} = 0.$$

This can be regarded as a system of simultaneous linear differential equations of the first order if $R_{3\tau 3\omega}$ satisfies (5). If $R_{3\sigma 12}$ and R_{1212} vanish on a surface $z^3 = f(z^1, z^2)$ in the shell, they vanish throughout the shell:

LEMMA 3. *When the conditions*

$$R_{3\sigma 3\omega} = 0 \qquad \textit{throughout a shell,} \tag{9}$$

$$R_{3\sigma 12} = R_{1212} = 0 \qquad \textit{on a surface } z^3 = f(z^1, z^2) \textit{ in the shell} \tag{10}$$

hold, it follows that

$$R_{rstu} = 0 \qquad \textit{throughout the shell.} \tag{11}$$

By virtue of LEMMA 3, the condition $R_{3\sigma 12} = 0$ on Π_+ can be derived from the condition $R_{3\sigma 12} = 0$ on Π_-, and *vice versa*. Furthermore, the conditions (5) and (6) can be replaced by

COROLLARY 4.1. *The condition of compatibility can be expressed as:*

$$R_{3\sigma 3\omega} = 0 \qquad \textit{throughout the shell} \tag{12}$$

$$R_{3\sigma 12} = R_{1212} = 0 \qquad \textit{on} \quad \Pi. \tag{13}$$

5. SIMPLIFICATION FOR THIN SHELLS

In this section it is assumed that the thickness of shells is small as compared with the radii of curvature:

$$th^\rho_\sigma \doteqdot 0 \qquad z^3 h^\sigma_\rho \doteqdot 0.$$

It then follows that

$$\Lambda^\sigma_\rho = A^{0i}_\rho A^\sigma_i \doteqdot \delta^\sigma_\rho \qquad \int \psi_{3\sigma;\rho}\, dz^3 \doteqdot \int \psi_{\tau\sigma} h^\tau_\rho\, dz^3 \doteqdot 0.$$

Now (22) and (23) of Section 3 become

$$\left.\begin{aligned} \Omega_{\rho\sigma}(P) &= \Psi_{\rho\sigma}(P_-) - \Psi_{\rho\sigma}(P_+) \\ \Omega_{\rho 3}(P) &= \Psi_{\rho 3}(P_-) - \Psi_{\rho 3}(P_-) \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} \Phi_\rho(P) &= \psi_\rho(P_-) + \frac{t}{2}\Psi_{3\rho}(P_-) - \psi_\rho(P_+) + \frac{t}{2}\Psi_{3\rho}(P_+) \\ \Phi_3(P) &= \psi_3(P_-) - \psi_3(P_+). \end{aligned}\right\} \tag{2}$$

Furthermore, functions defined on $\Pi_\pm$ such as $\Psi_{rs}(P_\pm)$ and $\psi_r(P_\pm)$ may be regarded as functions defined on Π;

$$\Psi_{rs}(P_\pm) \equiv \Psi_{\pm rs}(P) \qquad \psi_r(P_\pm) \equiv \psi_{\pm r}(P). \tag{3}$$

Accordingly, the derivatives of these quantities with respect to z^w can be defined by

$$\Psi_{rs/\!/\omega}(P_\pm) \equiv \Psi_{\pm rs/\!/\omega}(P) \qquad \psi_{r/\!/\omega}(P_\pm) \equiv \psi_{\pm r/\!/\omega}(P). \tag{4}$$

By virtue of (6) of Section 3, the following relations hold:

$$\left.\begin{aligned} &\psi_{\rho/\!/\tau}(\mathrm{P}_\pm) = \Psi_{\tau\rho}+\psi_{\tau\rho} \qquad \text{or} \qquad \Psi_{\tau\rho} = \psi_{[\rho/\!/\tau]} \\ &\psi_{3/\!/\tau}(\mathrm{P}_\pm) = \Psi_{\tau 3}. \end{aligned}\right\} \tag{5}$$

From (7) of Section 3 and (19) of Section 1, it is naturally assumed that

$$\psi_{\tau/\!/3} = -\Psi_{3/\!/\tau} = \Psi_{3\tau} \qquad \psi_{3/\!/3} = 0. \tag{6}$$

Since $\Psi_{3\rho/\!/\tau} = -\Psi_{3/\!/\rho\tau}$, it can be verified that

$$\Omega_{\tau\rho} = \Phi_{[\rho/\!/\tau]} \qquad \Omega_{r3} = \Phi_{3/\!/r} = -\Phi_{r/\!/3}. \tag{7}$$

By virtue of COROLLARY 2.1, there holds

THEOREM 5. *In the case of thin shells, stress resultants and couples are expressed by Φ_r alone:*

$$\left.\begin{aligned} N^{\rho\sigma} &= e^{\rho\tau}e^{\sigma\omega}\Phi_{3/\!/\tau\omega} \\ N^{3\sigma} &= e^{\rho\tau}e^{\sigma\omega}\tfrac{1}{2}\Phi_{\tau/\!/\rho\omega} \\ M^{\rho\sigma} &= e^{\rho\tau}e^{\sigma\omega}\Phi_{(\tau/\!/\omega)}. \end{aligned}\right\} \tag{8}$$

The function Φ_r itself is expressed in terms of ψ_r as:

$$\left.\begin{aligned} &\Phi_\rho(\mathrm{P}) = \psi_\rho(\mathrm{P}_-)-\psi_\rho(\mathrm{P}_+)-\frac{t}{2}\,(\psi_{3/\!/\rho}(\mathrm{P}_-)+\psi_{3/\!/\rho}(\mathrm{P}_+)) \\ &\Phi_3(\mathrm{P}) = \psi_3(\mathrm{P}_-)-\psi_3(\mathrm{P}_+). \end{aligned}\right\} \tag{9}$$

In many cases, $N^{3\sigma}$ and $M^{\rho\sigma}/t$ are small as compared with $N^{\rho\sigma}$. As can be seen from THEOREM 5, $N^{3\sigma}$ and $M^{\rho\sigma}$ depend principally upon Φ_ρ, and $N^{\rho\sigma}$ upon Φ_3. Therefore, Φ_ρ may be disregarded in comparison to Φ_3:

COROLLARY 5.1. *When $N^{3\sigma}$ and $M^{\rho\sigma}$ are small enough, Φ_ρ may be omitted as compared with Φ_3. In such a case, stress resultants and couples can be expressed in terms of Φ_3:*

$$\left.\begin{aligned} N^{\rho\sigma} &= e^{\rho\tau}e^{\sigma\omega}(\Phi_{3/\tau\omega}-\Phi_3 l_{\tau\omega}) \\ N^{3\sigma} &= e^{\rho\tau}e^{\sigma\omega}\Phi_{3/[\rho h\tau]\omega} \\ M^{\rho\sigma} &= e^{\rho\tau}e^{\sigma\omega}\Phi_3 h_{\tau\omega}. \end{aligned}\right\} \tag{10}$$

Since $\Phi_3 l_{\tau\omega}$ can generally be omitted in comparison to $\Phi_{3/\tau\omega}$, the expression for $N^{\rho\sigma}$ of (10) is equivalent to that in Donnell's theory for cylindrical shells [4].

CONCLUSION

In this paper, stress distributions in a shell not subjected to any external forces have been fully investigated in terms of stress function tensors. Stress resultants and stress couples are derived from the shell stress function of Gol'denveizer which is related to the stress function tensors. The corresponding condition of compatibility can be obtained by using the principle of complementary virtual work. The shell stress functions of other authors are derived from the stress function tensors.

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Résumé—Les contraintes dans une enveloppe peuvent être déduites des tenseurs de la fonction de tension qui sont considérés comme une extension de la fonction de tension Maxwell–Morera, tandis que les résultantes de tension et les couples de tension peuvent être déduits de la fonction de tension de Gol'denveizer. La relation entre les tenseurs de la fonction de tension et la fonction de tension des enveloppes est étudiée dans cet article.

Zusammenfassung—Beanspruchungen in einer Schale können von den Beanspruchungs-Tensoren hergeleitet werden, welche als eine Erweiterung der Maxwell–Morera Beanspruchungs Funktion angesehen werden können, während Beanspruchungsresultanten und Beanspruchungs Kupplungen von der Schalenbeanspruchungsfunktion von Gol'denveizer hergeleitet werden können. In dieser Abhandlung wird das Verhältniss zwischen den Funktionen Beanspruchungs Tensoren und der Hülsenbeanspruchungs Funktionen untersucht.

Абстракт—Напряжения в оболочке могут быть выведённы на тензоры функции напряжения, которые рассматриваются, как продолжение функции напряжения Максвелла–Морера, в то время, как равнодействующие напряжения и пары сил напряжения могут быть изолированы от функции напряжения оболочки Голденвейзера. В этой статье исследуется отношение между тензорами функции напряжения и функцией напряжения оболочки.